Optimization in modern power systems

Lecture 10: Semidefinite Programming and OPF

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The Goals for Today!

- Review of Day 9
- Reminder: Assignment 2
- Questions and Clarifications on Assignments
- Semidefinite Programming (SDP)
- Example in SDP
- Convex Relaxations for AC-OPF
- Presentation by Joachim and Ibrahim

Reviewing Day 9 in Groups!

- For 10 minutes discuss with the person sitting next to you about:
  - Three main points we discussed in yesterday's lecture
  - One topic or concept that is not so clear to you and you would like to hear again about it

Points you would like to discuss?

Questions about Assignments?

Assignment 2: Solution Methods

- Timeline
  - Jan 4 (today): handing out the assignment
  - Monday, Jan 16: peer-review process
  - Wednesday, Jan 18, 9am - 11am: Presentation in front of the class
- 15-minute presentation + 5 minutes questions
- Goal of the presentation. At the end of the presentation, the audience must be able to:
  - describe the basic principles of the solution method in 3 sentences
  - remember a key figure or a key equation that describes how the method works
  - list 2 advantages and 2 disadvantages of the presented method
- The presenting group and the peer-review group are expected to have an equally good knowledge of the subject. Questions can be addressed to both the presenting group and the group that reviewed it.

Peer-review process

- Two groups meet and review their presentations for two hours. During the first hour, the first group reviews the presentation of the second group. In the second hour, the presenting group becomes the peer-review group and vice versa.

- Goal of the peer-review process: the peer-review group must help the presenting group prepare a good presentation, that can be comprehensible from the rest of the class. During this process, the peer-review group is expected to gain a good understanding of the presentation topic (otherwise the peer-reviewing would not have been successful).

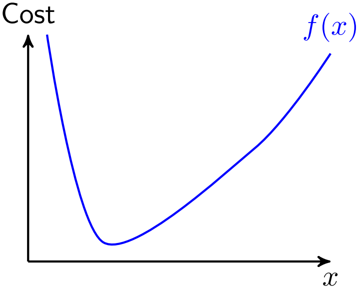
- You are free to spend as much time as you think necessary in peer-reviewing, but one hour per group is the minimum.

Outline of Lecture

- **Motivation: Convex vs. Non-Convex Problem and SDP**
- What is SDP?
  - Numerical Example
- What is a Positive Semidefinite Matrix?
- SDP vs. LP
- SDP Application on the MAX-CUT problem
- Convex Relaxations for AC-OPF

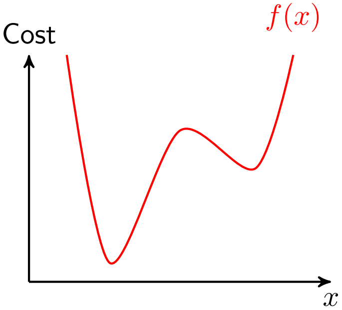
Convex vs. Non-convex Problem

Convex Problem



One global minimum

Non-convex problem

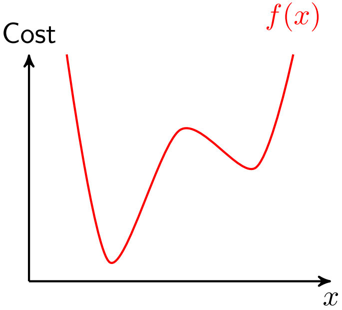


Several local minima

Several local minima: So what?

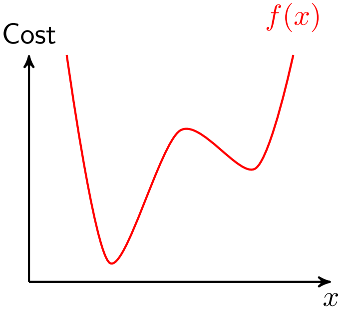
Example: Optimal Power Flow Problem

- Assume that the difference in the cost function of a local minimum versus a global minimum is 10%
- The total electric energy cost in the US is $\approx$ 400 Billion$/year
- 10% amounts to 40 billion US$ in economic losses per year
- Even 1% difference is huge
- Convex problems guarantee that we find a global minimum $\Rightarrow$ convexify the OPF problem



Convexifying the Optimal Power Flow problem (OPF)

- Convex relaxations transform the OPF to a convex Semi-Definite Program (SDP)



Convex Relaxation

[1] Javad Lavaei and Steven H Low. "Zero duality gap in optimal power flow problem". In: *IEEE Transactions on Power Systems* 27.1 (2012), pp. 92–107

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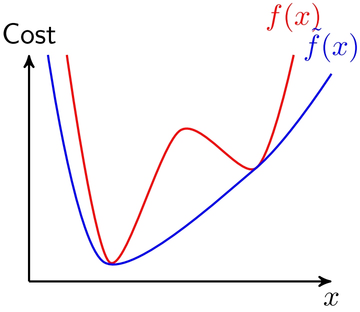

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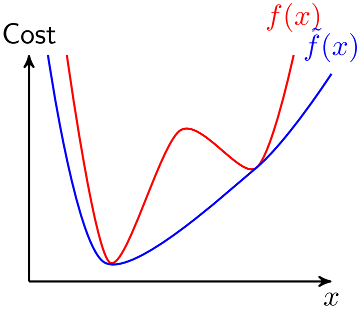
Convexifying the Optimal Power Flow problem (OPF)

- Convex relaxations transform the OPF to a convex Semi-Definite Program (SDP)
- Under certain conditions, the obtained solution is the global optimum to the original OPF problem[1]



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Semidefinite Programming

Semidefinite programming (SDP) is the most exciting development in the mathematical programming in the 1990's [2]

- Between 2008-2012 we had the first formulations (and breakthroughs) for a convexified AC-OPF problem.

[2] Robert M. Freund, Introduction to Mathematical Programming, MIT Lecture Notes, 2009

What is Semidefinite Programming? (SDP)

- SDP is the "generalized" form of an LP (linear program)

Linear Programming	Semidefinite Programming
$$\min c^T \cdot x$$	$$\min C \bullet X := \sum_i \sum_j C_{ij} X_{ij}$$
subject to:	subject to:
$$a_i \cdot x = b_i, \quad i = 1, \ldots, m$$ $$x \geq 0, \quad x \in R^n$$	$$A_i \bullet X = b_i, \quad i = 1, \ldots, m$$ $$X \succeq 0$$

- LP: Optimization variables in the form of a vector x.
- SDP: Optim. variables in the form of a positive semidefinite *matrix* X.

What is Semidefinite Programming? (SDP)

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- LP: Optimization variables in the form of a vector x.
- SDP: Optim. variables in the form of a positive semidefinite *matrix* X.

Positive Semidefinite Matrix??

Ignore it for now. We will come back to it in a few slides.

$C \bullet X := \sum_i \sum_j C_{ij} X_{ij}$ – What's that?

$C \bullet X$: "*sum* of elementwise multiplication"

$$\min \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

subject to:

$$\begin{bmatrix} A1_{11} & A1_{12} \\ A1_{21} & A1_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = b_1$$

$$\begin{bmatrix} A2_{11} & A2_{12} \\ A2_{21} & A2_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = b_1$$

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$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

min

$$c_{11} X_{11} + c_{12} X_{12} + c_{21} X_{21} + c_{22} X_{22}$$

subject to:

$$A1_{11} X_{11} + A1_{12} X_{12} + A1_{21} X_{21} + A1_{22} X_{22} = b_1$$

$$A2_{11} X_{11} + A2_{12} X_{12} + A2_{21} X_{21} + A2_{22} X_{22} = b_2$$

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

SDP vs LP

Semidefinite Programming

$$\min$$
$$c_{11}X_{11} + c_{12}X_{12} + c_{21}X_{21} + c_{22}X_{22}$$

subject to:

$$A1_{11}X_{11} + A1_{12}X_{12} + A1_{21}X_{21} + A1_{22}X_{22} = b_1$$
$$A2_{11}X_{11} + A2_{12}X_{12} + A2_{21}X_{21} + A2_{22}X_{22} = b_2$$
$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

Linear Program

In LP we have the optimization variables in a vector:

$$X = [X_{11}\ X_{12}\ X_{21}\ X_{22}]^T$$
$$\min c^T \cdot X$$

subject to:

$$A_1 \cdot X = b_1$$
$$A_2 \cdot X = b_2$$
$$X_{11} \geq 0,\ X_{12} \geq 0,$$
$$X_{21} \geq 0,\ X_{22} \geq 0$$

- SDP looks very much like a LP!
- Only difference: instead of each element of $\mathbf{X}$ to be positive, $\mathbf{X}$ must be a positive semidefinite matrix!

Numerical Example

- Assume X is a 3×3 matrix.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 7 \\ 1 & 7 & 5 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

$$b_1 = 11 b_2 = 19$$

Formulate the optimization problems w.r.t. to the elements of matrix X, i.e. linear equations w.r.t. X_{11}, X_{12}, etc.

Numerical Example

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Formulate the optimization problems w.r.t. to the elements of matrix X, i.e. linear equations w.r.t. X_{11}, X_{12}, etc.

Answer in p.6 of R. Freund, Introduction to Semidefinite Programming, MIT Lecture Notes, 2009. `https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-251j-introduction-to-mathematical-programming-fall-2009/readings/MIT6_251JF09_SDP.pdf`

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What is a Positive Semidefinite Matrix P?

- P must be symmetric

P is a positive semidefinite matrix iff:

- $x^T P x \geq 0$, for *any* non-zero vector x

  or

- $eig(P) \geq 0$ for *all* eigenvalues of P

  or

- all principal minors are non-negative

What are Principal Minors?

Principal minors are the determinants of submatrices of P

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \Rightarrow$$

first order: p_{22} second order: $\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}$

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \Rightarrow$$

first order: p_{22}

second order:

$$\begin{vmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{vmatrix} \quad \begin{vmatrix} p_{21} & p_{23} \\ p_{31} & p_{33} \end{vmatrix} \quad \begin{vmatrix} p_{21} & p_{22} \\ p_{31} & p_{32} \end{vmatrix}$$

third order: $\begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{vmatrix}$

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SDP vs LP

Does it make such a difference if we optimize over a positive semidefinite X instead of having all individual elements of this matrix positive?

Yes!

SDP vs LP variables: Example

$$X = \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0$$

When is P positive semidefinite?

SDP vs LP variables: Example

$$X = \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0$$

When is P positive semidefinite?

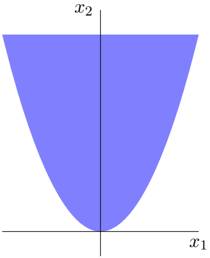
For X to be positive semidefinite, it must be:

- X symmetric $\rightarrow$ OK!
- first order princ.minor positive: $1 > 0 \rightarrow$ OK!
- second order princ.minor positive: $x_2 - x_1^2 \geq 0$

Example: Feasible space of SDP vs LP variables

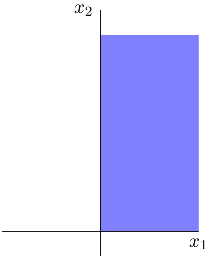
SDP

$$X = \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0 \Rightarrow x_2 - x_1^2 \geq 0$$



LP

$$x_1 \geq 0$$
$$x_2 \geq 0$$



Example: Feasible space of SDP vs LP variables

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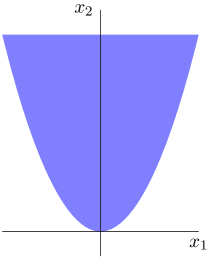
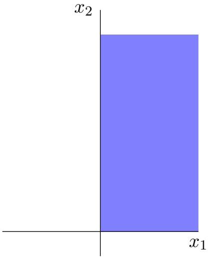


LP

$$x_1 \geq 0$$
$$x_2 \geq 0$$



- In SDP can express quadratic constraints, e.g. x_1^2 or $x_1 x_2$
- In general, in SDP we allow the variables to "move" in a larger space $\rightarrow$ here, x_1 can take negative values
- SDP applies to a larger family of problems $\rightarrow$ LP special case of SDP

LP as a special case of SDP

$$\min \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

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How should C, A_1, A_2 look like so that our SDP problem become an LP?

Assume that the LP will only have two variables.

LP as a special case of SDP

$$\min \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

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How should C, A_1, A_2 look like so that our SDP problem become an LP?

Assume that the LP will only have two variables.

Answer: If C, A_1, A_2 are diagonal, then our SDP problem is actually an LP!

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SDP Application – Preliminaries: Rank of a Matrix

- Assume matrix A with dimensions $M \times N = 5 \times 3$
- $0 \leq \text{rank}(A) \leq \min(M, N)$
- If all rows and columns are linearly independent, then $\text{rank}(A) = \min(M, N)$
  - If all rows and columns are linearly independent, how much is $\text{rank}(A)$, if A has dimensions 5×3?

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  - If all rows and columns are linearly independent, how much is $\text{rank}(A)$, if A has dimensions 5×3?
- It holds: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$
- B is a vector with dimension $N \times 1$
  - How much is $\text{rank}(B)$?
  - How much is $\text{rank}(AB)$?

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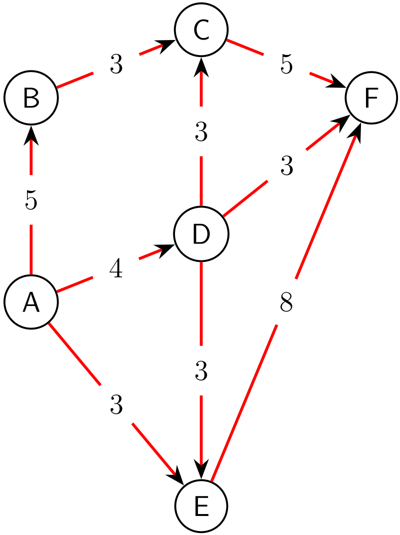
- B is a vector with dimension $N \times 1$
  - How much is $\text{rank}(B)$?
  - How much is $\text{rank}(AB)$?
- $W = XX^T$, where X is a vector. How much is $\text{rank}(W)$?

SDP Application: MAX-CUT Problem

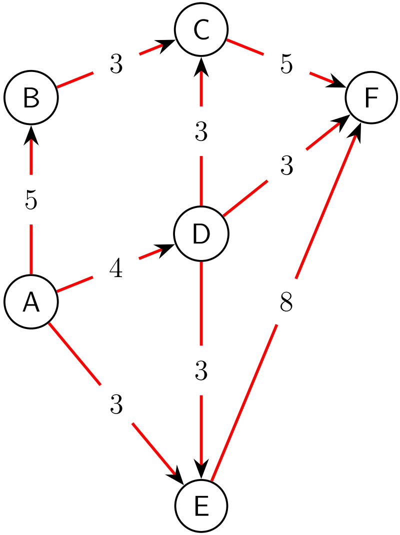
- Split the graph in two subgraphs: S and $\overline{S}$
- Goal: maximize the total weight of the edges between S and $\overline{S}$
- NP-complete problem: no fast/efficient solution is known
- Applications:
  - Data clustering: Split the data in two groups. Nearby data get clustered together. Data far away in opposite groups.
  - Maps: Identify the "borders" between two areas.
  - etc.



SDP Application: MAX-CUT Problem

What is the max-cut solution for this graph?

Split the graph in two subgraphs, so that the sum of the edge weights *between* the graphs becomes maximum.



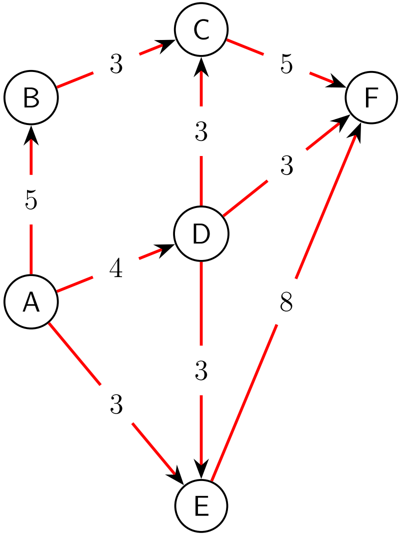
SDP Application: MAX-CUT Problem

$$\max \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(1 - x_i x_j)$$

subject to:

$$x_j \in \{-1, 1\}, j = 1, \ldots, n.$$

- Let $Y = xx^T$, where $Y_{ij} = x_i x_j$



$Y = xx^T$

$$Y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2x_2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3x_3 \end{bmatrix}$$

- Y captures all the squares, e.g. x_1^2, in the diagonal, and all the possible products between two vector elements in the off-diagonals.
- $Y \succeq 0$ by construction
- Y is a rank-1 matrix

SDP Application: MAX-CUT Problem

$$\max \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(1 - x_i x_j)$$

subject to:

$$x_j \in \{-1, 1\}, j = 1, \ldots, n.$$

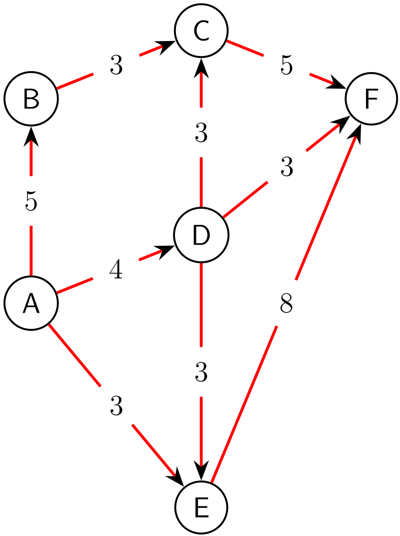
- Let $Y = xx^T$, where $Y_{ij} = x_i x_j$. Then:

$$\max \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} - W \bullet Y$$

subject to:

$$x_j \in \{-1, 1\}, j = 1, \ldots, n$$

$$Y = xx^T$$



SDP Application: MAX-CUT Problem

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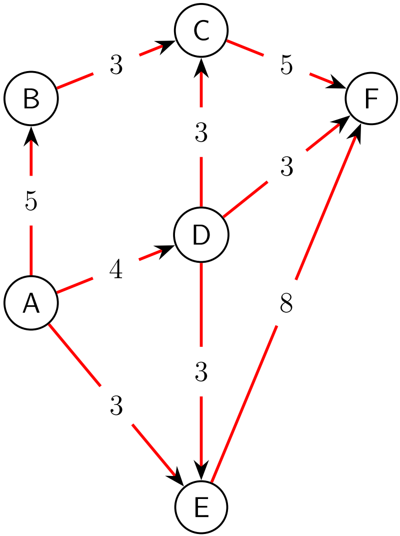
subject to:

$$x_j \in \{-1, 1\}, j = 1, \dots, n$$
$$Y = xx^T$$

$$\Downarrow$$

$$\max \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - W \bullet Y$$
$$Y_{jj} = 1, j = 1, \dots, n$$
$$Y = xx^T$$



SDP Application: MAX-CUT Problem

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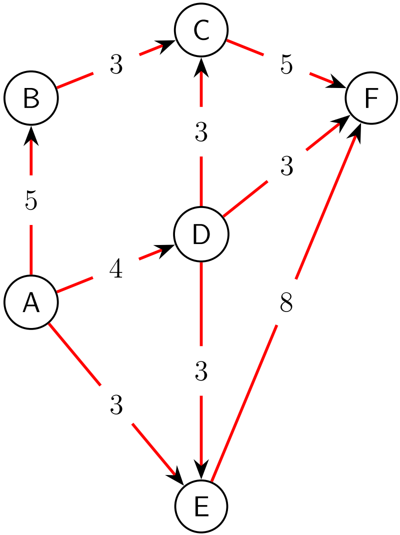
$$\Downarrow$$

$$\max \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - W \bullet Y$$
$$Y_{jj} = 1, j = 1, \dots, n$$
$$Y \succeq 0$$
$$\text{rank}(Y) = 1$$



SDP Application: MAX-CUT Problem

$$\max \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} - W \bullet Y$$

$$Y_{jj} = 1, j = 1, \ldots, n$$

$$Y = xx^T$$

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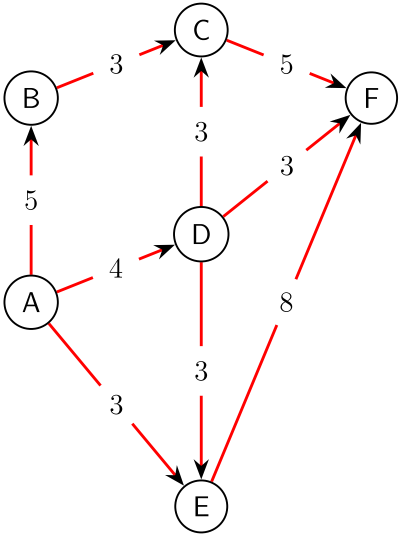
$$Y \succeq 0$$

~~$\text{rank}(Y) = 1$~~ Relax the problem!



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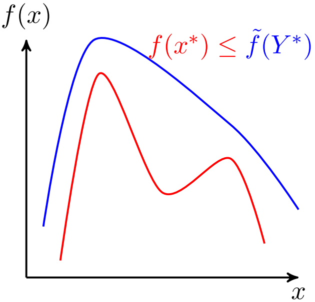
- For the objective functions, it holds EXACT $\leq$ RELAX
- The RELAX problem is an SDP problem!
- If the Y that we find happens also to be rank-1, then RELAX=EXACT!

SDP Application: MAX-CUT Problem



EXACT: $Y = xx^T$

$\Downarrow$

RELAX: $Y \succeq 0$

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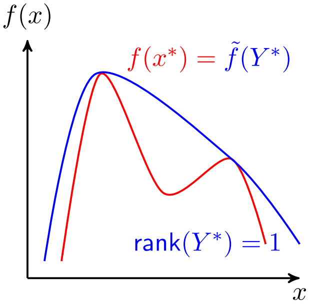

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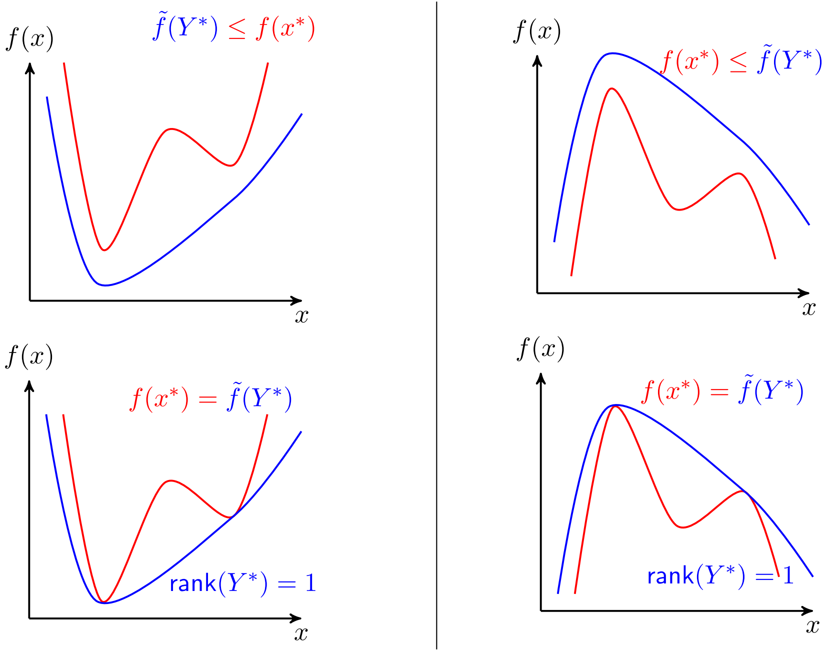
Minimization vs. Maximization



Outline of Lecture

- Motivation: Convex vs. Non-Convex Problem and SDP
- What is SDP?
  - Numerical Example
- What is a Positive Semidefinite Matrix?
- SDP vs. LP
- SDP Application on the MAX-CUT problem
- **Convex Relaxations for AC-OPF**

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- It has been shown that in most power systems the obtained W_{opt} is rank-1
- Still an open research topic!

Convex relaxations for AC-OPF

$$
\begin{aligned}
S_1 &= V_1 Y_{\text{bus}}^* V^* \\
&= V_1 (Y_{11} V_1 + Y_{12} V_2)^* \\
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\end{aligned}
$$

$$S_2 = Y_{22}^* V_2 V_2^* + Y_{21}^* V_2 V_1^*$$

- I define:

$$W = VV^H = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} V_1^* & V_2^* \end{bmatrix} = \begin{bmatrix} V_1 V_1^* & V_1 V_2^* \\ V_2 V_1^* & V_2 V_2^* \end{bmatrix}$$

It holds:

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It holds:

EXACT: $W \succeq 0$

rank(W) $= 1$

RELAX: $W \succeq 0$

~~rank(W) $= 1$~~

Wrap-up

- The SDP is a generalization of the LP
- The main difference between the formulation of the SDP and the LP, is that the SDP requires the variables to form a positive semidefinite matrix, while the LP requires all variables to be larger than zero.
- The SDP formulation allows for more “freedom” in the variables.
- SDP can model quadratic constraints.

Convex relaxations with SDP

- The initial problem has pairwise products of variables and square values, i.e, $x_i x_j, x_i^2$.
- Formulate a matrix $W = xx^T$. For W it holds $W \succeq 0$ and rank$(W) = 1$.
- rank$(W) = 1$ is not convex $\rightarrow$ drop it!
- The rest of the (relaxed) problem convex!
- Relaxed problem: contains *all* feasible solutions of the original problem, plus many more.
- Solve for W_{opt} and hope that rank$(W) = 1$ $\rightarrow$ then our solution is feasible for the original problem.
- If yes, then you just found the global optimum!
- Exactly this procedure is followed in the SDP formulation for the AC-OPF