Optimization in modern power systems

Lecture 5: Constrained Optimization, LMPs, and AC-OPF

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The Goals for Today!

- Review of Day 4
- Questions and Clarifications on Assignments
- Derivation of LMPs
- AC-OPF
- Y_{bus} and Y_{line}
- From the AC to the DC power flow equations

Reviewing Day 4 in Groups!

- For 10 minutes discuss with the person sitting next to you about:
  - Three main points we discussed in yesterday's lecture
  - One topic or concept that is not so clear to you and you would like to hear again about it

Points you would like to discuss?

Questions about the Assignments?

Notes on Assigment 1

- The built-in Matpower solver `MIPS` cannot find a solution for Case Study 5. Try a different solver:
  - e.g. MOSEK
  - other ?

Constrained Optimization: Example

$$\min_{x_1,x_2} (x_1 - 3)^2 + (x_2 - 2)^2$$

subject to:

$$\begin{aligned} 2x_1 + x_2 &= 8 \\ x_1 + x_2 &\leq 7 \\ x_1 - 0.25x_2^2 &\leq 0 \end{aligned}$$

- Write down the KKT conditions for this problem.

Constrained Optimization: Graphical Solution

Example:

$$\min_{x_1,x_2} (x_1-3)^2 + (x_2-2)^2$$

subject to:

$$\begin{aligned} 2x_1 + x_2 &= 8 \\ x_1 + x_2 &\leq 7 \\ x_1 - 0.25x_2^2 &\leq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

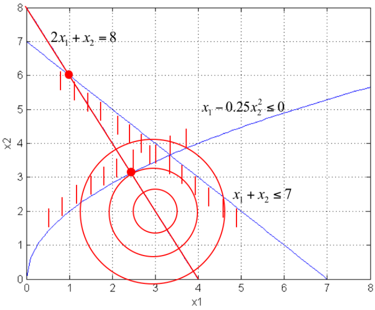


Figure taken from: Gabriela Hug, Lecture slides for class 18-879 M: Optimization in Energy Networks, Carnegie Mellon University, USA, 2015.

DC-OPF based on PTDF

$$\min \sum_{i=1}^{N_{P_G}} c_i P_{G,i},$$

subject to:

$$\sum_{i=1}^{N_{P_G}} P_{G,i} - \sum_{i=1}^{N_{P_L}} P_{L,i} = 0$$

$$-\mathbf{F_L} \leq \mathbf{PTDF} \cdot (\mathbf{P_G} - \mathbf{P_L}) \leq \mathbf{F_L}$$

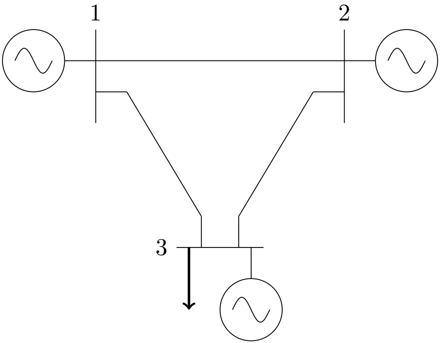
$$\mathbf{0} \leq \mathbf{P_G} \leq \mathbf{P_{G,max}}$$

Lagrangian of the DC-OPF

$$
\begin{aligned}
\mathcal{L}(P_G, \nu, \lambda, \mu) = \sum_{i=1}^{N_{P_G}} c_i P_{G,i} + \nu \cdot \left(\sum_{i=1}^{N_{P_G}} P_{G,i} - \sum_{i=1}^{N_{P_L}} P_{L,i} \right) \\
+ \sum_{i=1}^{N_L} \lambda_i^+ \cdot [\mathbf{PTDF_i} \cdot (\mathbf{P_G} - \mathbf{P_L}) - F_{L,i}] \\
+ \sum_{i=1}^{N_L} \lambda_i^- \cdot [-\mathbf{PTDF_i} \cdot (\mathbf{P_G} - \mathbf{P_L}) - F_{L,i}] \\
+ \sum_{i=1}^{N_{P_G}} \mu_i^+ \cdot (P_{G,i} - P_{G,i,max}) + \sum_{i=1}^{N_{P_G}} \mu_i^- \cdot (-P_{G,i})
\end{aligned}
$$

Test System

- Assume a 3-bus system with 3 generators, and 1 load on bus 3
- We assume an auxilliary variable ξ_3 that represents very small changes of the load in Bus 3. We assume $\xi_3 = 0$.
- Then it is $\hat{P}_L = P_L + \Xi$, where $\Xi = [0\ 0\ \xi_3]^T$.



Lagrangian of the DC-OPF with Ξ

$$\mathcal{L}(P_G, \nu, \lambda, \mu, \Xi) = \sum_{i=1}^{N_{P_G}} c_i P_{G,i} + \nu \cdot \left(\sum_{i=1}^{N_{P_G}} P_{G,i} - \sum_{i=1}^{N_{P_L}} P_{L,i} - \xi_i \right)$$

$$+ \sum_{i=1}^{N_L} \lambda_i^+ \cdot [\mathbf{PTDF_i} \cdot (\mathbf{P_G} - \mathbf{P_L} - \Xi) - F_{L,i}]$$

$$+ \sum_{i=1}^{N_L} \lambda_i^- \cdot [-\mathbf{PTDF_i} \cdot (\mathbf{P_G} - \mathbf{P_L} - \Xi) - F_{L,i}]$$

$$+ \sum_{i=1}^{N_{P_G}} \mu_i^+ \cdot (P_{G,i} - P_{G,i,max}) + \sum_{i=1}^{N_{P_G}} \mu_i^- \cdot (-P_{G,i}).$$

Lagrangian of DC-OPF for the 3-bus system

- To save space in this slide: $K_i \equiv PTDF_i$

$$\mathcal{L}(P_G, \nu, \lambda, \mu, \xi_3) = \sum_{i=1}^{N_{P_G}} c_i P_{G,i} + \nu \cdot \left(\sum_{i=1}^{N_{P_G}} P_{G,i} - \sum_{i=1}^{N_{P_L}} P_{L,i} - \xi_3 \right)$$

$$+ \sum_{i=1}^{N_L} \lambda_i^+ \cdot [K_{i,1} \cdot P_{G,1} + K_{i,2} \cdot P_{G,2} + K_{i,3} \cdot (P_{G,3} - P_{L,3} - \xi_3) - F_{L,i}]$$

$$+ \sum_{i=1}^{N_L} \lambda_i^- \cdot [-K_{i,1} \cdot P_{G,1} - K_{i,2} \cdot P_{G,2} - K_{i,3} \cdot (P_{G,3} - P_{L,3} - \xi_3) - F_{L,i}]$$

$$+ \sum_{i=1}^{N_{P_G}} \mu_i^+ \cdot (P_{G,i} - P_{G,i,max}) + \sum_{i=1}^{N_{P_G}} \mu_i^- \cdot (-P_{G,i}).$$

KKTs for the DC-OPF: No congestion

- No congestion $\Rightarrow$ all $\lambda_i = 0$
- One marginal generator: only one generator has both $\mu_i^+ = 0$ and $\mu_i^- = 0$
- Assume G2 is marginal; $P_{G1} = P_{G1,max}$; $P_{G3} = 0$.

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$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^+ = 0$$

$$c_2 + \nu = 0$$

$$c_3 + \nu + \mu_3^- = 0$$

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$$c_1 + \nu + \mu_1^+ = 0$$
$$c_2 + \nu = 0$$
$$c_3 + \nu + \mu_3^- = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu$$

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$$c_1 + \nu + \mu_1^+ = 0$$
$$c_2 + \nu = 0$$
$$c_3 + \nu + \mu_3^- = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu$$

Attention! ξ_3 does not exist in the optimization problem and is not an optimization variable. We do not need to derive any KKT conditions w.r.t. ξ_3, e.g. $\frac{\partial L}{\partial \xi_3} = 0$.

ξ_3 is just an auxilliary variable. It helps us "represent" the marginal change in the load of bus 3. $\frac{\partial L}{\partial \xi_3}$ quantifies its effect on the Lagrangian.

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$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^+ = 0$$
$$c_2 + \nu = 0$$
$$c_3 + \nu + \mu_3^- = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu$$

$LMP_3 = -\nu = c_2$: nodal price on bus 3!

How much is the LMP on the other buses?

KKTs for the DC-OPF: One congested line

- Assume that line 1-3 gets congested in the direction $1 \rightarrow 3 \Rightarrow \lambda_{13}^{+} \neq 0$
- Now G2 and G3 are both marginal gens; $P_{G1} = P_{G1,max}$.

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$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^{+} + \lambda_{13}^{+} PTDF_{13,1} = 0$$

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$

$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

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$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu - \lambda_{13}^{+} PTDF_{13,3}$$

KKTs for the DC-OPF: One congested line

- Assume that line 1-3 gets congested in the direction $1 \rightarrow 3 \Rightarrow \lambda_{13}^{+} \neq 0$
- Now G2 and G3 are both marginal gens; $P_{G1} = P_{G1,max}$.

$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^{+} + \lambda_{13}^{+} PTDF_{13,1} = 0$$

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$

$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu - \lambda_{13}^{+} PTDF_{13,3}$$

To find LMP_3 I need ν and λ_{13}^{+}

How do I find ν and λ_{13}^{+}?

KKTs for the DC-OPF: One congested line

- Solve the linear system with 2 equations and 2 unknowns: ν and λ_{13}^{+}

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$
$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

KKTs for the DC-OPF: One congested line

- Solve the linear system with 2 equations and 2 unknowns: ν and λ_{13}^{+}

$$
\begin{aligned}
c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0 \\
c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0
\end{aligned}
$$

- What can we say about the LMPs on different buses?

$$LMP_i = -\nu - \lambda_{13}^{+} PTDF_{13,i}$$

KKTs for the DC-OPF: One congested line

- Solve the linear system with 2 equations and 2 unknowns: ν and λ_{13}^{+}

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$
$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

- What can we say about the LMPs on different buses?

$$LMP_i = -\nu - \lambda_{13}^{+} PTDF_{13,i}$$

- If there is a congestion, the LMPs are no longer the same on every bus. They are dependent on the congestion!

AC-OPF

- Minimize

- subject to:

AC-OPF

- Minimize

Costs, Line Losses, other?

- subject to:

AC-OPF

- Minimize

Costs, Line Losses, other?

- subject to:

AC Power Flow equations

Line Flow Constraints

Generator Active Power Limits

Generator Reactive Power Limits

Voltage Magnitude Limits

(Voltage Angle limits to improve solvability)

(maybe other equipment constraints)

AC-OPF

- Minimize

Costs, Line Losses, other?

- subject to:

AC Power Flow equations

Line Flow Constraints → Line Current Limits; Apparent Power Flow limits; Active Power Flow limits

Generator Active Power Limits

Generator Reactive Power Limits

Voltage Magnitude Limits

(Voltage Angle limits to improve solvability)

(maybe other equipment constraints)

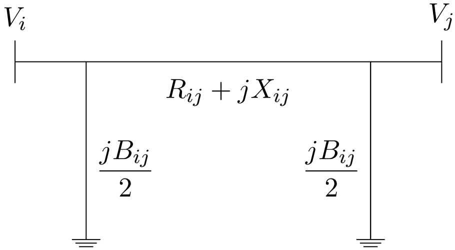
- Optimization vector: $[P\ Q\ V\ \theta]^T$

AC-OPF[1]

obj.function	$\min c^T P_G$
AC flow	$S_G - S_L = diag(\overline{V})\overline{Y}^*_{\text{bus}}\overline{V}^*$
Line Current	$\lvert\overline{Y}_{\text{line},i\rightarrow j}\overline{V}\rvert \leq I_{line,max}$
	$\lvert\overline{Y}_{\text{line},j\rightarrow i}\overline{V}\rvert \leq I_{line,max}$
or Apparent Flow	$\lvert\overline{V}_i\overline{Y}^*_{\text{line},i\rightarrow j,\text{i-row}}\overline{V}^*\rvert \leq S_{i\rightarrow j,max}$
	$\lvert\overline{V}_j\overline{Y}^*_{\text{line},j\rightarrow i,\text{j-row}}\overline{V}^*\rvert \leq S_{j\rightarrow i,max}$
Gen. Active Power	$0 \leq P_G \leq P_{G,max}$
Gen. Reactive Power	$-Q_{G,max} \leq Q_G \leq Q_{G,max}$
Voltage Magnituge	$V_{min} \leq V \leq V_{max}$
Voltage Magnituge	$V_{min} \leq V \leq V_{max}$
Voltage Angle	$\theta_{min} \leq \theta \leq \theta_{max}$

[1] All shown variables are vectors or matrices. The bar above a variable denotes complex numbers. $(\cdot)^*$ denotes the complex conjugate. To simplify notation, the bar denoting a complex number is dropped in the following slides. Attention! The *current* flow constraints are defined as *vectors*, i.e. for all lines. The apparent power *line* constraints are defined *per line*.

Current flow along a line



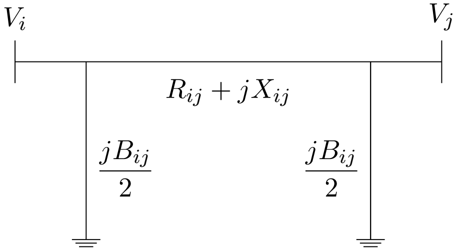
π-model of the line

It is:

$$y_{ij} = \frac{1}{R_{ij} + jX_{ij}}$$

$y_{sh,i} = j\frac{B_{ij}}{2} +$ other shunt elements connected to that bus

Current flow along a line



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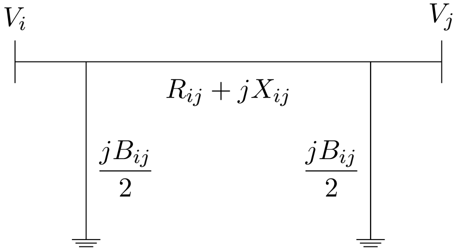
$$y_{ij} = \frac{1}{R_{ij} + jX_{ij}}$$

$y_{sh,i} = j\frac{B_{ij}}{2} +$ other shunt elements connected to that bus

$$i \rightarrow j: \quad I_{i \rightarrow j} = y_{sh,i} V_i + y_{ij}(V_i - V_j) \Rightarrow I_{i \rightarrow j} = \begin{bmatrix} y_{sh,i} + y_{ij} & -y_{ij} \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix}$$

Current flow along a line



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$$i \rightarrow j: \quad I_{i\rightarrow j} = y_{sh,i}V_i + y_{ij}(V_i - V_j) \Rightarrow I_{i\rightarrow j} = \begin{bmatrix} y_{sh,i} + y_{ij} & -y_{ij} \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix}$$

$$j \rightarrow i: \quad I_{j\rightarrow i} = y_{sh,j}V_j + y_{ij}(V_j - V_i) \Rightarrow I_{j\rightarrow i} = \begin{bmatrix} -y_{ij} & y_{sh,j} + y_{ij} \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix}$$

Line Admittance Matrix Y_{line}

- $\mathbf{Y}_{\text{line}}$ is an $L \times N$ matrix, where L is the number of lines and N is the number of nodes

- if row k corresponds to line $i - j$:
  - $Y_{\text{line},ki} = y_{sh,i} + y_{ij}$
  - $Y_{\text{line},kj} = -y_{ij}$

- $y_{ij} = \dfrac{1}{R_{ij} + jX_{ij}}$ is the admittance of line ij

- $y_{sh,i}$ is the shunt capacitance $jB_{ij}/2$ of the π-model of the line

- We must create two Y_{line} matrices. One for $i \rightarrow j$ and one for $j \rightarrow i$

Bus Admittance Matrix $Y_{\textsf{bus}}$

$$S_i = V_i I_i^*$$

$$I_i = \sum_k I_{ik}, \text{ where } k \text{ are all the buses connected to bus } i$$

Example: Assume there is a line between nodes $i-m$, and $i-n$. It is:

$$\begin{aligned} I_i &= I_{im} + I_{in} \\ &= (y_{sh,i}^{i\rightarrow m} + y_{im})V_i - y_{im}V_m + (y_{sh,i}^{i\rightarrow n} + y_{in})V_i - y_{in}V_n \\ &= (y_{sh,i}^{i\rightarrow m} + y_{im} + y_{sh,i}^{i\rightarrow n} + y_{in})V_i - y_{im}V_m - y_{in}V_n \end{aligned}$$

$$I_i = [\underbrace{y_{sh,im} + y_{im} + y_{sh,in} + y_{in}}_{Y_{\text{bus},ii}} \ \underbrace{-y_{im}}_{Y_{\text{bus},im}} \ \underbrace{-y_{in}}_{Y_{\text{bus},in}}][V_i \ V_m \ V_n]^T$$

Bus Admittance Matrix Y_{bus}

- $\mathbf{Y}_{\text{bus}}$ is an $N \times N$ matrix, where N is the number of nodes
- diagonal elements: $Y_{\text{bus},ii} = y_{sh,i} + \sum_k y_{ik}$, where k are all the buses connected to bus i
- off-diagonal elements:
  - $Y_{\text{bus},ij} = -y_{ij}$ if nodes i and j are connected by a line[2]
  - $Y_{\text{bus},ij} = 0$ if nodes i and j are not connected
- $y_{ij} = \dfrac{1}{R_{ij} + jX_{ij}}$ is the admittance of line ij
- $y_{sh,i}$ are all shunt elements connected to bus i, including the shunt capacitance of the π-model of the line

[2] If there are more than one lines connecting the same nodes, then they must all be added to $Y_{\text{bus},ij}, Y_{\text{bus},ii}, Y_{\text{bus},jj}$.

AC Power Flow Equations

$$S_i = V_i I_i^* = V_i Y_{\text{bus}}^* V^*$$

For all buses $S = [S_1 \dots S_N]^T$:

$$S_{\text{gen}} - S_{\text{load}} = diag(V) Y_{\text{bus}}^* V^*$$

From AC to DC Power Flow Equations

- The power flow along a line is:

$$S_{ij} = V_i I_{ij}^* = V_i(y_{sh,i}^* V_i^* + y_{ij}^*(V_i^* - V_j^*))$$

- Assume a negligible shunt conductance: $g_{sh,ij} = 0 \Rightarrow y_{sh,i} = jb_{sh,i}$.
- Given that $R << X$ in transmission systems, for the DC power flow we assume that $z_{ij} = r_{ij} + jx_{ij} \approx jx_{ij}$. Then $y_{ij} = -j\frac{1}{x_{ij}}$.
- Assume: $V_i = V_i\angle 0$ and $V_j = V_j\angle\delta$, with $\delta = \theta_j - \theta_i$.

$$\begin{aligned} I_{ij}^* &= -jb_{sh,i}V_i + j\frac{1}{x_{ij}}(V_i - (V_j\cos\delta - jV_j\sin\delta)) \\ &= -jb_{sh,i}V_i + j\frac{1}{x_{ij}}V_i - j\frac{1}{x_{ij}}V_j\cos\delta - \frac{1}{x_{ij}}V_j\sin\delta \end{aligned}$$

From AC to DC Power Flow Equations (cont.)

- Since V_i is a real number, it is:

$$P_{ij} = \Re\{S_{ij}\} = V_i \Re\{I_{ij}^*\} = -\frac{1}{x_{ij}} V_i V_j \sin\delta$$

- With $\delta = \theta_j - \theta_i$, it is:

$$P_{ij} = \frac{1}{x_{ij}} V_i V_j \sin(\theta_i - \theta_j)$$

- We further make the assumptions that:
  - V_i, V_j are constant and equal to 1 p.u.
  - $\sin\theta \approx \theta$, θ must be in rad

Then

$$P_{ij} = \frac{1}{x_{ij}} (\theta_i - \theta_j)$$