Optimization in modern power systems

Lecture 8: Duality

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The Goals for Today!

- Review of Day 7
- Questions and Clarifications on Assignments
- Duality
- Duality in LP

Schedule for the rest of the course

- Tomorrow: Lecture 9am-10am only
- Fri, Jan 15: Convex AC-OPF: Semidefinite programming
- Mon, Jan 16: Peer-review of Assignment 2 (R113 & R153 booked)
- Tue, Jan 17: Repetition – prepare questions!
- Wed, Jan 18: Presentation of Assignment 2
- Thu, Jan 19: no lecture, 1pm-3pm questions in this room (B325-R113)
- Fri, Jan 20: Exam
- Mon, Jan 23: Deadline for Assignments 1 and 3

Reviewing Day 7 in Groups!

- For 10 minutes discuss with the person sitting next to you about:
  - Three main points we discussed in yesterday's lecture
  - One topic or concept that is not so clear to you and you would like to hear again about it

Points you would like to discuss?

Questions about the Assignments?

Dual Problem

With the help of the Lagrangian function and the Lagrangian multipliers, we can define and solve a dual optimization problem.

- Primal problem: our original problem
- Dual problem: the problem we formulate with the help of the Lagrangian
- Dual variables $\equiv$ Lagrangian multipliers

Why do we care about the dual?

Advantages of the dual problem:

- it might be easier to solve, e.g. less constraints
- always concave → convex optimization
- always gives a lower bound to the objective value of our original problem
- for certain set of problems, e.g. convex → exact
  - Strong duality → The dual problem of convex primal problems *usually* results to the same solution as the primal problem

The dual function is concave

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) = \inf_{x \in D} \left(f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x) \right)$$

- $\inf_{x \in D}$ stands for the minimum value of the Lagrangian over x: for $\lambda \in R^m, \nu \in R^p$
- g is always concave: Lagrangian is linear with respect to λ, ν and $\inf$ preserves concavity
- The dual function is concave, even if f_0, f_i, h_i are non-convex/non-concave.

The dual function is concave: Example

$$\min x_1^2 + x_2^2$$

subject to:

$$x_1 + x_2 - 4 = 0$$

Find the dual function $g(\nu) = \inf_{x \in D} L(x, \nu)$

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Find the dual function $g(\nu) = \inf_{x \in D} L(x, \nu)$

$$L(x, \nu) = x_1^2 + x_2^2 + \nu(x_1 + x_2 - 4)$$
$$g(\nu) = \inf_{x \in D} L(x, \nu) \Rightarrow \nabla_x L = 0$$

$$\nabla_x L = \begin{bmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + \nu \\ 2x_2 + \nu \end{bmatrix} = 0 \Rightarrow \begin{matrix} x_1 = -\frac{\nu}{2} \\ x_2 = -\frac{\nu}{2} \end{matrix}$$

$$L(\nu) = -\frac{\nu^2}{2} - 4\nu \Rightarrow \text{concave!}$$

Dual function: lower bound

- For *any* $\lambda \geq 0$ and *any* ν, it holds:

$$g(\lambda, \nu) \leq f_0(x^*)$$

- Assume $\tilde{x}$ feasible point, i.e. $f_i(\tilde{x}) \leq 0,\ h_i(\tilde{x}) = 0,\ \lambda \geq 0$. Then we have

$$\begin{aligned}
\sum_i \lambda_i f_i(\tilde{x}) + \sum_i \nu_i h_i(\tilde{x}) \leq 0 \\
L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_i \lambda_i f_i(\tilde{x}) + \sum_i \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x}) \\
g(\lambda, \nu) = \inf_{x \in D} L(\tilde{x}, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})
\end{aligned}$$

- This holds for every feasible point $\tilde{x}$, including the optimal point x^*.

Strong and weak duality

- Dual problem:

$$\max g(\lambda, \nu)$$
$$\text{subject to } \lambda \geq 0$$

- Always a convex problem!
- Weak duality: $g(\lambda^*, \nu^*) \leq f_0(x^*)$
- Strong duality: $g(\lambda^*, \nu^*) = f_0(x^*)$
- Duality gap: $g(\lambda^*, \nu^*) - f_0(x^*)$
- Strong duality usually holds for convex problems!

- Dual: convex & lower bound $\Rightarrow$ Cheap certificate!
- If $g(\lambda^*, \nu^*) = f_0(x^*)$, it's guaranteed that this is the global optimum

Strong duality: example

$$\min x_1^2 + x_2^2$$

subject to:

$$x_1 + x_2 - 4 = 0$$

Dual:

$$L(\nu) = -\frac{\nu^2}{2} - 4\nu$$

1. Find $\min_x f_0(x)$ s.t. $h(x) = 0$
2. Find $\max_\nu L(\nu)$

- What do you observe?
- Which problem is it easier to solve?

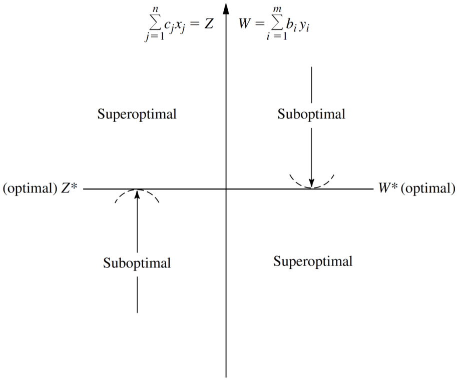
Dual of a Linear Program

LP in standard form		Dual Problem
$$\min c^T x$$ subject to $Ax = b$, $x \geq 0$	$\Leftrightarrow$	$$\max \; -b^T \nu$$ subject to $A^T \nu + c \geq 0$
minimize $\#n$ variables x $\#p$ equality constraints	$\Leftrightarrow$	maximize $\#n$ inequality constraints $\#p$ dual variables ν

- $-\nu^T b = -\nu^T A x \leq c^T x$: if x and ν are feasible solutions, $-b^T \nu \leq c^T x$.
- if x^* and ν^* are feasible solutions and $-b^T \nu = c^T x$, then x^* and ν^* are the optimal solutions for their respective problems.

Two different paths with the same endpoint

Dual problem   Primal Problem



Slide inspired from Juan-Miguel Morales, 02435 Decision-Making under uncertainty in Electricity Markets, DTU.

Figure taken from: F.S. Hillier, G.J. Liebermann. Introduction to Operations Research. McGraw Hill, 2001.

Strong duality and KKT conditions

$$L(x, \lambda, \nu) = f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x) \tag{1}$$

Strong duality: When does $L(x^*, \lambda, \nu) = f_0(x^*)$ hold?

Remember:

$h_i(x) = 0$

$f_i(x) \leq 0$

$\lambda_i \geq 0$

Strong duality and KKT conditions

$$L(x, \lambda, \nu) = f_0(x) + \sum_i \lambda_i f_i(x) + \sum_i \nu_i h_i(x) \quad (1)$$

Strong duality: When does $L(x^*, \lambda, \nu) = f_0(x^*)$ hold?

Remember:

$h_i(x) = 0$

$f_i(x) \leq 0$

$\lambda_i \geq 0$

When $\lambda_i f_i(x^*) = 0$

(complementary slackness)

KKT conditions hold only if strong duality exists

- KKT conditions require that $\lambda_i f_i(x^*) = 0$

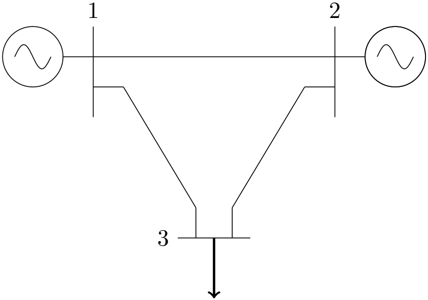
KKT conditions hold only in case of strong duality

- Strong duality *usually* holds in convex problems $\Rightarrow$ DC-OPF is convex
- Convex problems with strong duality: KKTs are necessary and sufficient.

Convex problems (such as DC-OPF):
If any point satisfies the KKT conditions, then it is the global optimal.

- We can solve either the primal or the dual problem: same objective value at x^*, due to strong duality

Question: What is the dual of the DC-OPF?



$$\min c_1 P_{G1} + c_2 P_{G2}$$

subject to:

$$B\theta = P_G - P_L$$

$$P_G \geq 0$$

- no line flow constraints

Duality: Wrap-up

- The dual problem is a convex optimization problem
- Lower bound and weak duality: if x^* and λ^*, ν^* feasible, then $g(\lambda^*, \nu^*) \leq f_0(x^*)$
- Strong duality: if x^* and λ^*, ν^* feasible solutions and $g(\lambda^*, \nu^*) = f_0(x^*)$, then x^* and λ^*, ν^* are the optimal solutions for their respective problems.
- If dual unbounded above, the primal is infeasible – and vice versa: if primal unbounded below, the dual is infeasible.
- The dual can provide a cheap certificate for a lower bound of the objective value.
- In general if the primal has more constraints than variables, the dual will have more variables than constraints:
  - less constraints $\rightarrow$ easier to solve