Optimization in modern power systems

Lecture 4: Lagrangian and Nodal Prices

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DTU Electrical Engineering
Department of Electrical Engineering

Groups and Topics for Assignment 2

1. Primal-dual interior-point method:
2. Simplex method:
3. Newton's method for optimization with equality constraints:
4. Gradient descent method for unconstrained optimization:

- Peer-review groups
  - #1 with #3
  - #2 with #4

The Goals for Today!

- Review of Day 3
- Questions and Clarifications on Assignments
- Lagrangian for Inequality Constrained Optimization
- Extracting the Lagrangian Multipliers (= nodal prices) for the DC-OPF problem

Reviewing Day 3 in Groups!

- For 10 minutes discuss with the person sitting next to you about:
  - Three main points we discussed in yesterday's lecture
  - One topic or concept that is not so clear to you and you would like to hear again about it

Points you would like to discuss?

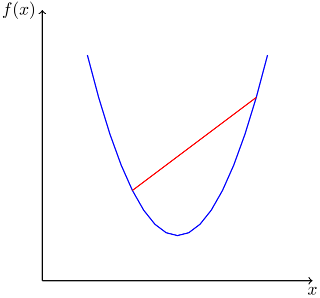
Questions about the Assignments?

Notes on Assigment 1

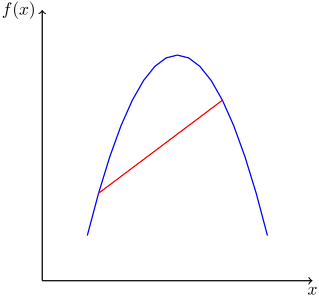
- The line flow constraints of the DC-OPF must be considered for both directions

Convex and Concave Functions

- Convex function: a line connecting two points must lie above the function



- Concave function: a line connecting two points must lie below the function



- Ideally, we want to minimize convex functions and maximize concave functions

Formulating an optimization problem

Example: James, a CMU student, opens a new sandwich shop on CMU campus to earn some money. He offers two types of sandwiches, tuna and chicken. His costs for the tuna sandwich are $4, his profit is $3.5 and it takes him 8 minutes to make one. The costs for the chicken sandwich are $6, his profit is $3 and it takes him 6 minutes to make one. Besides studying, he is able to spend 3 hours per day preparing sandwiches and he has a budget of $120 per day. The university regulations say that he has to sell at least 5 sandwiches of each type.

- Assuming that James can sell all his sandwiches, write down the optimization problem to find the number of sandwiches of each type which maximize his profit.

Formulating an optimization problem

Example: James, a CMU student, opens a new sandwich shop on CMU campus to earn some money. He offers two types of sandwiches, tuna and chicken. His costs for the tuna sandwich are $4, his profit is $3.5 and it takes him 8 minutes to make one. The costs for the chicken sandwich are $6, his profit is $3 and it takes him 6 minutes to make one. Besides studying, he is able to spend 3 hours per day preparing sandwiches and he has a budget of $120 per day. The university regulations say that he has to sell at least 5 sandwiches of each type.

- Assuming that James can sell all his sandwiches, write down the optimization problem to find the number of sandwiches of each type which maximize his profit.
- Answer: 15 tuna sandwiches, 10 chicken sandwiches

Formulating an optimization problem

Example: James, a CMU student, opens a new sandwich shop on CMU campus to earn some money. He offers two types of sandwiches, tuna and chicken...

- Assuming that James can sell all his sandwiches, write down the optimization problem to find the number of sandwiches of each type which maximize his profit.
- Answer: 15 tuna sandwiches, 10 chicken sandwiches
- Note that this is normally a mixed integer linear problem (MILP). In our case, we *relax* our problem and assume that the optimization variables are continuous variables. This allows us to solve it with `linprog`. We were "lucky" and the solver returned integers as the optimal result. If we did not obtain integers, our solution would have been *infeasible* for the original problem. Then we would need to use different methods to solve it, e.g. using the `intlinprog` from the Matlab Optimization Toolbox.

Equality Constrained Optimization

- Example:

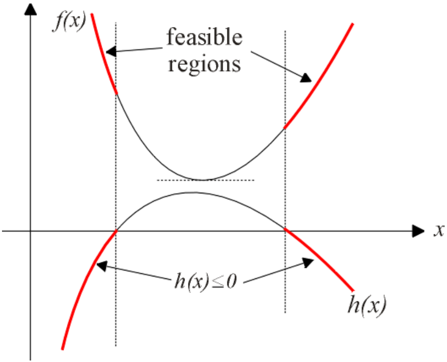
$$\min_{x} x_1^2 + x_2^2$$
$$s.t. -x_1 - x_2 + 4 = 0$$

- Find the solution to this problem using KKT conditions.

Inequality Constrained Optimization

- Find a solution to:

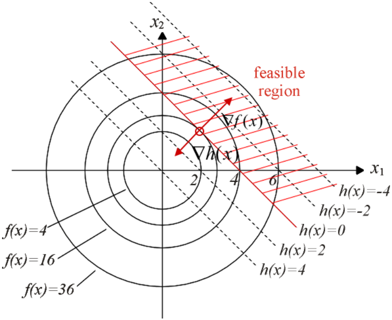
$$\min_{x} f_0(x)$$
$$\text{s.t.} \quad f_i(x) = 0 \quad \text{for } i=1,\ldots,m$$



$$\frac{\partial f_0(x)}{\partial x} \neq 0 \text{ in feasible region}$$

Binding Constraint

- Constraint binding, i.e. $f_i(x^*) = 0$



$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$$

$$f_i \leq 0 \quad \text{for } i = 1, \ldots, m$$

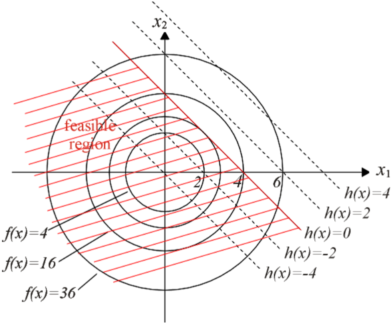
- gradients of objective function and of constraint are in opposite directions in optimal point

$$\Rightarrow \lambda_i > 0$$

- Sensitivity: $\lambda_i = -\dfrac{\Delta f_0(x)}{\Delta f_i(x)}$

Non-binding constraint

- Constraint non-binding, i.e. $f_i(x^*) < 0$



$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$$

$$f_i \leq 0 \quad \text{for } i = 1, \ldots, m$$

- gradient of objective function is zero, i.e. $\nabla f(x) = 0$

$$\Rightarrow \lambda_i = 0$$

- The Lagrange multiplier is:
  - $>$0, for binding inequality constraints
  - $=$0, for non-binding inequality constraints

KKTs for Inequality Constrained Optimization

- Lagrange function

$$L = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$$

- Minimize Lagrange function

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f(x)}{\partial x} + \sum_{i=1}^{m} \lambda_i \frac{\partial f_i(x)}{\partial x} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \Rightarrow f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m$$

$$\lambda_i f_i(x) = 0 \quad \text{for } i + 1, \ldots, m$$

$$\lambda_i \geq 0$$

Solution can be found by checking combinations of binding and non-binding constraints $\Rightarrow$ use solution algorithms

KKTs for Constrained Optimization

- Minimize Lagrange function:

$$L = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- The Karush-Kuhn-Tacker first order or necessary optimality conditions:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f(x)}{\partial x} + \sum_{i=1}^{m} \lambda_i \frac{\partial f_i(x)}{\partial x} + \sum_{i=1}^{p} \nu_i \frac{\partial h_i(x)}{\partial x} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \Rightarrow f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m$$

$$\frac{\partial L}{\partial \nu_i} = 0 \Rightarrow h_i(x) = 0 \quad \text{for } i = 1, \ldots, p$$

$$\lambda_i f_i(x) = 0 \quad \text{for } i + 1, \ldots, m$$

$$\lambda_i \geq 0$$

Costrained Optimization: Example

$$\min_{x_1,x_2} (x_1-3)^2 + (x_2-2)^2$$

subject to:

$$\begin{aligned} 2x_1 + x_2 &= 8 \\ x_1 + x_2 &\leq 7 \\ x_1 - 0.25x_2^2 &\leq 0 \end{aligned}$$

- Write down the KKT conditions for this problem.

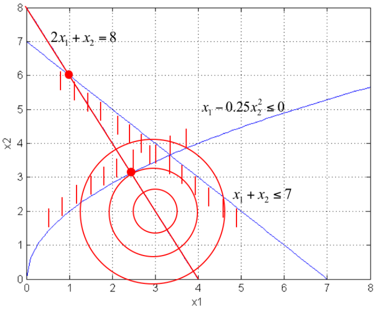
Constrained Optimization: Graphical Solution

Example:

$$\min_{x_1,x_2} (x_1 - 3)^2 + (x_2 - 2)^2$$

subject to:

$$2x_1 + x_2 = 8$$
$$x_1 + x_2 \leq 7$$
$$x_1 - 0.25x_2^2 \leq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



Mathematical Formulations: Summary

Unconstrained Optimization

$$\min_x f(x)$$

Equality Constrained Optimization

$$\min_x f(x)$$

$$h_i(x) = 0 \quad \text{for } i = 1, \ldots, p$$

Inequality Constrained Optimization

$$\min_x f(x)$$

$$f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m$$

General Constrained Optimization

$$\min_x f(x)$$

$$f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, p$$

$$h_i(x) = 0 \quad \text{for } i = 1, \ldots, m$$

Convex Optimization

- The optimization problem

$$\min_x f(x)$$
$$f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, p$$
$$h_i(x) = 0 \quad \text{for } i = 1, \ldots, m$$

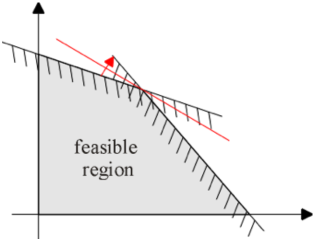
is convex if:

- the objective function $f(x)$ is convex
- the inequality constraints $f_i(x)$ are convex
- the equality constraints $h_i(x)$ are linear

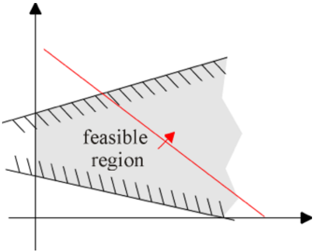
If the problem is convex, there is a single optimum, which is also the global optimum!

Solution Types for Linear Optimization

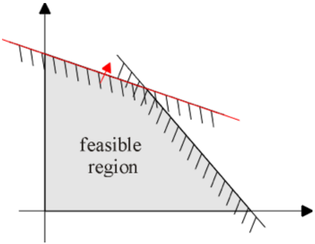
Unique Solution



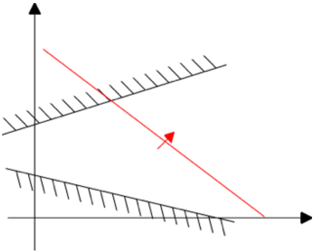
Unbounded Solution



Infinitely many solutions



No solution



DC-OPF based on PTDF

$$\min \sum_{i=1}^{N_{P_G}} c_i P_{G,i},$$

subject to:

$$\sum_{i=1}^{N_{P_G}} P_{G,i} - \sum_{i=1}^{N_{P_L}} P_{L,i} = 0$$

$$-\mathbf{F_L} \leq \mathbf{PTDF} \cdot (\mathbf{P_G} - \mathbf{P_L}) \leq \mathbf{F_L}$$

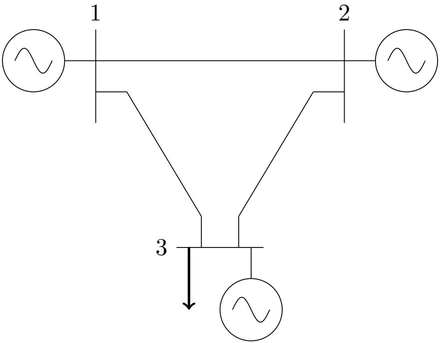
$$\mathbf{0} \leq \mathbf{P_G} \leq \mathbf{P_{G,max}}$$

Lagrangian of the DC-OPF

$$
\begin{aligned}
\mathcal{L}(P_G, \nu, \lambda, \mu) = \sum_{i=1}^{N_{P_G}} c_i P_{G,i} + \nu \cdot \left(\sum_{i=1}^{N_{P_G}} P_{G,i} - \sum_{i=1}^{N_{P_L}} P_{L,i} \right) \\
+ \sum_{i=1}^{N_L} \lambda_i^+ \cdot [\mathbf{PTDF_i} \cdot (\mathbf{P_G} - \mathbf{P_L}) - F_{L,i}] \\
+ \sum_{i=1}^{N_L} \lambda_i^- \cdot [-\mathbf{PTDF_i} \cdot (\mathbf{P_G} - \mathbf{P_L}) - F_{L,i}] \\
+ \sum_{i=1}^{N_{P_G}} \mu_i^+ \cdot (P_{G,i} - P_{G,i,max}) + \sum_{i=1}^{N_{P_G}} \mu_i^- \cdot (-P_{G,i})
\end{aligned}
$$

Test System

- Assume a 3-bus system with 3 generators, and 1 load on bus 3
- We assume an auxilliary variable ξ_3 that represents very small changes of the load in Bus 3. We assume $\xi_3 = 0$.
- Then it is $\hat{P}_L = P_L + \Xi$, where $\Xi = [0\ 0\ \xi_3]^T$.



Lagrangian of the DC-OPF with Ξ

$$\mathcal{L}(P_G, \nu, \lambda, \mu, \Xi) = \sum_{i=1}^{N_{P_G}} c_i P_{G,i} + \nu \cdot \left(\sum_{i=1}^{N_{P_G}} P_{G,i} - \sum_{i=1}^{N_{P_L}} P_{L,i} - \xi_i \right)$$

$$+ \sum_{i=1}^{N_L} \lambda_i^+ \cdot [\mathbf{PTDF_i} \cdot (\mathbf{P_G} - \mathbf{P_L} - \Xi) - F_{L,i}]$$

$$+ \sum_{i=1}^{N_L} \lambda_i^- \cdot [-\mathbf{PTDF_i} \cdot (\mathbf{P_G} - \mathbf{P_L} - \Xi) - F_{L,i}]$$

$$+ \sum_{i=1}^{N_{P_G}} \mu_i^+ \cdot (P_{G,i} - P_{G,i,max}) + \sum_{i=1}^{N_{P_G}} \mu_i^- \cdot (-P_{G,i}).$$

Lagrangian of DC-OPF for the 3-bus system

- To save space in this slide: $K_i \equiv PTDF_i$

$$\mathcal{L}(P_G, \nu, \lambda, \mu, \xi_3) = \sum_{i=1}^{N_{P_G}} c_i P_{G,i} + \nu \cdot \left(\sum_{i=1}^{N_{P_G}} P_{G,i} - \sum_{i=1}^{N_{P_L}} P_{L,i} - \xi_3 \right)$$

$$+ \sum_{i=1}^{N_L} \lambda_i^+ \cdot [K_{i,1} \cdot P_{G,1} + K_{i,2} \cdot P_{G,2} + K_{i,3} \cdot (P_{G,3} - P_{L,3} - \xi_3) - F_{L,i}]$$

$$+ \sum_{i=1}^{N_L} \lambda_i^- \cdot [-K_{i,1} \cdot P_{G,1} - K_{i,2} \cdot P_{G,2} - K_{i,3} \cdot (P_{G,3} - P_{L,3} - \xi_3) - F_{L,i}]$$

$$+ \sum_{i=1}^{N_{P_G}} \mu_i^+ \cdot (P_{G,i} - P_{G,i,max}) + \sum_{i=1}^{N_{P_G}} \mu_i^- \cdot (-P_{G,i}).$$

KKTs for the DC-OPF: No congestion

- No congestion $\Rightarrow$ all $\lambda_i = 0$
- One marginal generator: only one generator has both $\mu_i^+ = 0$ and $\mu_i^- = 0$
- Assume G2 is marginal; $P_{G1} = P_{G1,max}$; $P_{G3} = 0$.

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$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^+ = 0$$

$$c_2 + \nu = 0$$

$$c_3 + \nu + \mu_3^- = 0$$

KKTs for the DC-OPF: No congestion

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$$c_1 + \nu + \mu_1^+ = 0$$
$$c_2 + \nu = 0$$
$$c_3 + \nu + \mu_3^- = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu$$

KKTs for the DC-OPF: No congestion

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$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^+ = 0$$
$$c_2 + \nu = 0$$
$$c_3 + \nu + \mu_3^- = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu$$

Attention! ξ_3 does not exist in the optimization problem and is not an optimization variable. We do not need to derive any KKT conditions w.r.t. ξ_3, e.g. $\frac{\partial L}{\partial \xi_3} = 0$.

ξ_3 is just an auxilliary variable. It helps us "represent" the marginal change in the load of bus 3. $\frac{\partial L}{\partial \xi_3}$ quantifies its effect on the Lagrangian.

KKTs for the DC-OPF: No congestion

- No congestion $\Rightarrow$ all $\lambda_i = 0$
- One marginal generator: only one generator has both $\mu_i^+ = 0$ and $\mu_i^- = 0$
- Assume G2 is marginal; $P_{G1} = P_{G1,max}$; $P_{G3} = 0$.

$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^+ = 0$$
$$c_2 + \nu = 0$$
$$c_3 + \nu + \mu_3^- = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu$$

$LMP_3 = -\nu = c_2$: nodal price on bus 3!

How much is the LMP on the other buses?

KKTs for the DC-OPF: One congested line

- Assume that line 1-3 gets congested in the direction $1 \rightarrow 3 \Rightarrow \lambda_{13}^{+} \neq 0$
- Now G2 and G3 are both marginal gens; $P_{G1} = P_{G1,max}$.

KKTs for the DC-OPF: One congested line

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- Now G2 and G3 are both marginal gens; $P_{G1} = P_{G1,max}$.

$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^{+} + \lambda_{13}^{+} PTDF_{13,1} = 0$$

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$

$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

KKTs for the DC-OPF: One congested line

- Assume that line 1-3 gets congested in the direction $1 \rightarrow 3 \Rightarrow \lambda_{13}^{+} \neq 0$
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$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^{+} + \lambda_{13}^{+} PTDF_{13,1} = 0$$

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$

$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu - \lambda_{13}^{+} PTDF_{13,3}$$

KKTs for the DC-OPF: One congested line

- Assume that line 1-3 gets congested in the direction $1 \rightarrow 3 \Rightarrow \lambda_{13}^{+} \neq 0$
- Now G2 and G3 are both marginal gens; $P_{G1} = P_{G1,max}$.

$$\frac{\partial \mathcal{L}}{\partial P_{G,i}} = 0, \quad \text{for all } i \in N_{P_G}$$

$$c_1 + \nu + \mu_1^{+} + \lambda_{13}^{+} PTDF_{13,1} = 0$$

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$

$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

Marginal change in the cost function for a marginal change in load:

$$LMP_3 = \frac{\partial \mathcal{L}}{\partial \xi_3} = -\nu - \lambda_{13}^{+} PTDF_{13,3}$$

To find LMP_3 I need ν and λ_{13}^{+}
How do I find ν and λ_{13}^{+}?

KKTs for the DC-OPF: One congested line

- Solve the linear system with 2 equations and 2 unknowns: ν and λ_{13}^{+}

$$
\begin{aligned}
c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0 \\
c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0
\end{aligned}
$$

KKTs for the DC-OPF: One congested line

- Solve the linear system with 2 equations and 2 unknowns: ν and λ_{13}^{+}

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$
$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

- What can we say about the LMPs on different buses?

$$LMP_i = -\nu - \lambda_{13}^{+} PTDF_{13,i}$$

KKTs for the DC-OPF: One congested line

- Solve the linear system with 2 equations and 2 unknowns: ν and λ_{13}^{+}

$$c_2 + \nu + \lambda_{13}^{+} PTDF_{13,2} = 0$$
$$c_3 + \nu + \lambda_{13}^{+} PTDF_{13,3} = 0$$

- What can we say about the LMPs on different buses?

$$LMP_i = -\nu - \lambda_{13}^{+} PTDF_{13,i}$$

- If there is a congestion, the LMPs are no longer the same on every bus. They are dependent on the congestion!